Int. J. Solids Structures Vol. 15, pp. 589–605
Pergamon Press Ltd., 1979. Printed in Great Britain

A FINITE ELEMENT APPROACH FOR LARGE STRAINS OF NEARLY INCOMPRESSIBLE RUBBER-LIKE MATERIALS

S. Cescotto† and G. Fonder‡
Service de Résistance des Matériaux et Statique des Constructions, Université de Liège, Liège, Belgium

(*Received* 13 *November* 1978)

Abstract—In [7], Nagtegaal *et al.* developed a variational principle for nearly incompressible materials in the fully plastic range. Here, this approach is extended to the domain of nonlinear elasticity for hyperelastic, isotropic, nearly incompressible bodies undergoing large strains. A very general form of the constitutive law is introduced into an extended functional, the connexion of which is established with Hellinger-Reissner principle. Similarly, extended forms of Key's [23] and Herrmann's [2] functionals are also found. Then, it is shown that the introduction of "modified invariants" of the metric tensor enables us to write this functional in a remarkably simple form. Finally, the proposed formulation is illustrated by a numerical example.

1. INTRODUCTION

Since the pioneer work of Herrmann [1–3] a considerable amount of effort has been devoted to the numerical solution of problems involving incompressible or nearly incompressible materials.

1.1 *State of the art in linear analysis*

Except for oriented bodies such as beams, plane stress membranes, thin shells..., for which no special difficulty arises from the incompressibility owing to the fact that some components of the stress tensor are assumed to be zero in a particular direction, several special methods have been proposed in the frame of the linear analysis by finite elements. They may be classified as follows [4].

(a) *Methods based on the discretization of the displacement field.* It is well known that the finite element analysis of nearly incompressible materials by displacement models may produce catastrophic results. The works of Fried [5, 6], Nagtegaal *et al.* [7], Argyris *et al.* [8–10] and Debongnie [11, 12] have provided a good understanding of this phenomenon. However, in some cases, a good solution can be obtained by using the reduced or selective numerical integration [13, 14] of the element stiffness matrix. This has been noticed by Zienkiewicz and Godbole [15], Naylor [16] and explained by Fried [6, 17]. More recently, Malkus and Hughes [18–22] have developed an equivalence theorem showing that the reduced or selective integration in displacement-type finite elements for nearly incompressible materials is equivalent to the use of a mixed model based on Herrmann's principle which will be recalled hereafter (see Section 2).

(b) *Method based on the discretization of the stress field.* It is easy to understand that a finite element of equilibrium type is able to treat a nearly incompressible material without special difficulty, because the incompressibility condition will be satisfied only in the mean sense. However, for a perfectly incompressible material, it can be shown [4] that the flexibility matrix is singular and this fact prevents the use of this kind of element when Poisson's coefficient $\nu = 0.5$.

(c) *Mixed formulations based on the Hellinger–Reissner's principle.* For an incompressible material, the hydrostatic stress becomes an independent variable. Thus, it can be discretized independently of the displacement field. This idea has been applied by Herrmann [2], Key [23] and Nagtegaal *et al.* [7]. They have proposed three different variational principles which are closely related to the Hellinger-Reissner principle. In these formulations, the discretization of the hydrostatic stress is very important and can lead to a complete or an average incompressibility of the finite element [24, 25]. In the first case, monotonic convergence of the deformation energy is obtained while this property is lost in the second case. However, the relaxation of the strict incompressibility condition makes the element more flexible and produces better results at a lower computer cost.

†Assistant.
‡Associate Professor.

(d) *Hybrid formulation based on Pian's principle.* For nearly incompressible materials, the assumed stress hybrid finite elements[26] may be used without special difficulty, as for the pure stress models. But, as they fail when $\nu = 0.5$, P. Tong[27] has modified the formulation for the perfectly incompressible case. It can be shown[47] that this modification also holds for the pure stress models.

1.2 *State of the art in nonlinear analysis*

In the nonlinear domain, we will restrict our attention to hyperelastic rubber-like materials. The volume changes accompanying the deformations of such bodies are very small[28] and it is convenient to treat them as incompressible or nearly incompressible solids. For these materials, the available formulations may be summarized as follows.

(a) *Oden's method.* Oden[29] suggests to apply the minimum energy principle in which the internal energy W per unit undeformed volume should be expressed by

$$W = \hat{W}(I_1, I_2) + \frac{p}{2}(I_3 - 1) \tag{1}$$

where I_1, I_2, I_3 are the three invariants of the metric tensor, and p is an hydrostatic pressure that should be discretized independently of the displacement field. This approach produces an additional set of pressure-like unknowns. Consequently, an additional set of equations has to be found. This is made by expressing that the deformation of the solid must be isochoric. The method has been described and applied in[e.g. 29–33].

(b) *Scharnhorst and Pian's method.* In[34], a modification of the assumed stress hybrid finite elements developed in[35] is proposed. In this approach, three fields are separately discretized: the displacements, the hydrostatic pressure and that part of the stress tensor deriving from the isochoric deformation potential $\hat{W}(I_1, I_2)$. The discretization parameters of the latter field are eliminated at the element level so that only displacement and pressure-like variables are retained in the final system of equations. In[34, 36], the method is applied to the finite element analysis of a hollow cylinder under internal pressure, in plane state of strain.

(c) *Formulations for nearly incompressible material.* In[37], Skala proposes to treat the rubber-like solids as nearly incompressible bodies the constitutive equation of which being

$$W = C_1(I_1 - I_3 - 2) + C_3(\sqrt{I_3} - 1)^2. \tag{2}$$

Then, an eight nodes isoparametric plane strain element based on the minimum energy principle is developed and applied to the same numerical example as in[34, 36]. Although the reduced numerical integration technique, as suggested by Naylor[16] was used, the author noticed "some oscillation in the calculated stresses as the material becomes more incompressible".

In[44], Hughes *et al.* use as constitutive law

$$W = \frac{1}{2}\lambda(\ln J)^2 + rE_{ij}E_{ij} \tag{3}$$

where λ and r are material parameters and $J = \sqrt{I_3}$.

For nearly incompressible materials, a large value of λ is adopted. Generalizing the idea of Fried[6] to the nonlinear case, selective numerical integration is used to compute the tangent stiffness matrix. This anticipates the extension of the equivalence theorem of Malkus and Hughes[22] to the large deformation domain.

(d) *Oriented bodies.* The remarks formulated in the linear case remain valid in nonlinear elasticity. Theoretical developments as well as practical applications may be found in the works of Oden *et al.*[e.g. 29, 45, 46], Bathe *et al.*[47, 48], Leonard and Verma[49] and many others.

1.3 *Proposed improvement*

In this paper, we propose an extension of Nagtegaal's *et al.* variational principle for the case of hyperelastic rubber-like solids. Some aspects of this approach are related to the above formulations for nearly incompressible materials. However, the present developments are more general because no restriction is imposed on the constitutive law (except that the material should be incompressible or nearly incompressible) and a total freedom is retained to satisfy exactly or approximately the incompressibility condition. The connection with Hellinger-Reissner's principle is established and it is shown that the introduction of "modified invariants" leads to a remarkably simple formulation.

2. MIXED PRINCIPLES IN LINEAR INCOMPRESSIBLE OR NEARLY INCOMPRESSIBLE ELASTICITY

2.1 *Herrmann's principle*[2]

Consider the functional

$$H(\sigma, u_i) = \int_V \left\{\frac{1}{2} 2G\epsilon_{ij}\epsilon_{ij} + \frac{3\nu}{1+\nu}\sigma\partial_i u_i - \frac{9\nu(1-2\nu)}{2(1+\nu)E}\sigma^2 - \rho g_i u_i\right\} \mathrm{d}V - \int_{A_T} t_i u_i \,\mathrm{d}A_T \qquad (4)$$

where u_i are the displacements, ϵ_{ij} the infinitesimal strains computed from the displacement field, σ is an independent pressure-like variable, G the shear modulus, ν Poisson's coefficient, E Young's modulus, ρ the mass density, g_i the distributed body force per unit mass, t_i the surface traction, V the volume of the body and A_T that part of the external area subjected to prescribed surface forces; ∂_i denotes the partial derivative with respect to the Cartesian coordinate x_i.

Expressing that this functional is stationary, the following Euler equations are obtained:

$$\partial_i u_i = \frac{3(1-2\nu)}{E}\sigma \qquad (5)$$

$$-\frac{3\nu}{1+\nu}\partial_j(\sigma\delta_{ij}) - \partial_j(2G\epsilon_{ij}) = \rho g_i. \qquad (6)$$

The introduction of (5) in (6) clearly gives the classical equilibrium equation.

2.2 *Key's principle*[23]

The functional proposed by Key writes, in the isotropic case

$$H(\sigma, u_i) = \int_V \left\{\sigma\partial_i u_i - \frac{1}{2}\frac{3(1-2\nu)}{E}\sigma^2 + \frac{1}{2}2G\hat{\epsilon}_{ij}\hat{\epsilon}_{ij} - \rho g_i u_i\right\} \mathrm{d}V - \int_{A_T} t_i u_i \,\mathrm{d}A_T \qquad (7)$$

where $\hat{\epsilon}_{ij}$ is the deviatoric strain:

$$\hat{\epsilon}_{ij} = \epsilon_{ij} - \frac{1}{3}\epsilon_{ll}\delta_{ij}. \qquad (8)$$

Euler equations of Key's functional are:

$$\partial_i u_i = \frac{3(1-2\nu)}{E}\sigma \qquad (9)$$

$$-\partial_j(\sigma\delta_{ij}) - \partial_j(2G\hat{\epsilon}_{ij}) = \rho g_i. \qquad (10)$$

2.3 *Nagtegaal's principle*[7]

The functionals recalled above are valid for all physically admissible values of ν, including $\nu = 0.5$. But, for a compressible or nearly incompressible material, eqn (9) can be inverted. In this case, the volumetric dilatation

$$\theta = \frac{3(1-2\nu)}{E}\sigma \qquad (11)$$

may be discretized instead of σ.

Introducing (11) in (7), Nagtegaal *et al.* have obtained the socalled Nagtegaal's functional:

$$H(\theta, u_i) = \int_V \left\{\chi\left(\theta\partial_i u_i - \frac{1}{2}\theta^2\right) + \frac{1}{2}2G\hat{\epsilon}_{ij}\hat{\epsilon}_{ij} - \rho g_i u_i\right\} \mathrm{d}V - \int_{A_T} t_i u_i \, \mathrm{d}A_T, \tag{12}$$

the Euler equations of which are:

$$\theta = \partial_i u_i \tag{13}$$

$$-\partial_j(\chi\theta\delta_{ij}) - \partial_j(2G\hat{\epsilon}_{ij}) = \rho g_i \tag{14}$$

where

$$\chi = \frac{E}{3(1-2\nu)} \tag{15}$$

is the bulk modulus.

2.4 *An intermediate principle* [4]

For compressible or nearly incompressible materials, the functionals of Key and Nagtegaal *et al.* are perfectly equivalent. As for the former, it differs from Herrmann's by the fact that the term $2G\hat{\epsilon}_{ij}\hat{\epsilon}_{ij}$ is purely deviatoric, while $2G\epsilon_{ij}\epsilon_{ij}$ still contains a small energy associated with volume change. This appears clearly in the equality

$$\frac{1}{2}2G\epsilon_{ij}\epsilon_{ij} = \frac{1}{2}2G\hat{\epsilon}_{ij}\hat{\epsilon}_{ij} + \frac{1-2\nu}{1+\nu}\sigma\delta_{ij}\epsilon_{ij} - \frac{1}{2}\frac{1-2\nu}{1+\nu}\frac{\sigma^2}{\chi}. \tag{16}$$

It is easy to introduce an intermediate principle in which a variable part of dilatational energy is uncoupled from the deviatoric energy. The following decomposition is performed

$$\hat{\epsilon}_{ij} = \alpha\hat{\epsilon}_{ij} + (1-\alpha)\left(\epsilon_{ij} - \frac{1-2\nu}{E}\sigma\delta_{ij}\right) \tag{17}$$

where α is an arbitrary coefficient. The use of this equality in (7) gives

$$H(\sigma, u_i) = \int_V \left\{\frac{1}{2}2G[(2\alpha-\alpha^2)\hat{\epsilon}_{ij}\hat{\epsilon}_{ij} + (1-\alpha)^2\epsilon_{ij}\epsilon_{ij}] + \left[1-(1-\alpha)^2\frac{1-2\nu}{1+\nu}\right]\right.$$
$$\left.\times\left[\sigma\partial_i u_i - \frac{1}{2}\frac{\sigma^2}{\chi}\right] - \rho g_i u_i\right\} \mathrm{d}V - \int_{A_T} t_i u_i \, \mathrm{d}A_T. \tag{18}$$

One can verify that for $\alpha = 0$ and 1, Herrmann's and Key's approaches are respectively recovered.

Besides its generality, this intermediate principle enables to switch easily from one to the other formulation as will be seen hereafter. In [4], this method has been tried in the particular case of a hollow sphere under intermal pressure, with $\alpha = 0$; $\alpha = 0.5$; $\alpha = 1$. The results do not differ significantly and agree very well with the analytical solution.

2.5 *Nearly incompressible materials*

For the sake of simplicity, assume that, in (18)

$$\begin{gathered}\sigma = \sigma_0 = \text{constant in } V \\ \nu \text{ and } \chi \text{ are also constant in } V.\end{gathered} \tag{19}$$

The variation on σ_0 gives

$$\psi = \frac{1}{V}\int_V \partial_i u_i \, \mathrm{d}V = \frac{\sigma_0}{\chi}. \tag{20}$$

If χ is finite, (20) can be inverted and the result introduced in (18). This produces the single field functional

$$H(u_i)=\int_V\left\{W(\epsilon_{ij})+\frac{1}{2}\chi\left[1-(1-\alpha)^2\frac{1-2\nu}{1+\nu}\right]\right.$$
$$\left.\times\left[\left(\frac{1}{V}\int_V\partial_i u_i\,\mathrm{d}V\right)^2-\epsilon_{ll}\epsilon_{rr}\right]-\rho g_i u_i\right\}\mathrm{d}V-\int_{A_T}t_i u_i\,\mathrm{d}A_T \tag{21}$$

where

$$W(\epsilon_{ij})=\frac{1}{2}2G\hat{\epsilon}_{ij}\hat{\epsilon}_{ij}+\frac{1}{2}\chi_{ll}\epsilon_{rr} \tag{22}$$

is the classical strain energy.

The question arises whether it is possible to put (21) in the form

$$H(u_i)=\int_V(W(\overset{*}{\epsilon}_{ij})-\rho g_i u_i)\,\mathrm{d}V-\int_{A_T}t_i u_i\,\mathrm{d}A_T \tag{23}$$

where

$$\overset{*}{\epsilon}_{ij}=\epsilon_{ij}+\frac{1}{3}\delta_{ij}F(\Psi,\epsilon_{rr})$$

are modified strains. A similar transformation has been performed by Nagtegaal *et al.*[7] for their principle. Here the possibility of its extension to the intermediate principle is examined.

Developing (23) and comparing with (21), it is found that $\overset{*}{\epsilon}_{ij}$ should write

$$\overset{*}{\epsilon}_{ij}=\epsilon_{ij}+\frac{1}{3}\delta_{ij}\,[\sqrt{[(1-m)\epsilon_{11}\epsilon_{rr}+m\Psi^2]}-\epsilon_{ll}]$$

with

$$m=1-(1-\alpha)^2\frac{1-2\nu}{1+\nu}$$

This shows that the only interesting case is $m=1(\alpha=0)$; this is precisely the case of Nagtegaal's principle

$$\overset{*}{\epsilon}_{ij}=\epsilon_{ij}+\frac{1}{3}\delta_{ij}(\Psi-\epsilon_{11}). \tag{24}$$

For $\alpha\neq 0$, such a simplification does not occur; consequently the use of modified strains cannot be generalized.

This makes Nagtegaal's functional especially attractive. Furthermore, the independent discretization of the volumetric dilatation θ enables to enforce average incompressibility at the finite element level with the most adequate number of constraints (see[17] for more details). These are the reasons why the extension of this approach to the nonlinear elasticity is examined hereafter.

3. A CONSTITUTIVE EQUATION FOR A NEARLY INCOMPRESSIBLE HYPERELASTIC MATERIAL

3.1 *An approximate constitutive equation*

In general, rubber-like materials are considered isotropic, hyperelastic and incompressible. Obviously, these characteristics are an idealization of the real behaviour of commercial rubbers[40]. But these properties may be used as a definition of an ideal rubber-like material[39] and in this case, the constitutive equation is expressed by means of an isochoric deformation potential $\hat{W}$ depending either on the invariants I_1, I_2 of the metric tensor or on the principal stretches λ_1, λ_2, λ_3 related by the incompressibility condition

$$\lambda_1\cdot\lambda_2\cdot\lambda_3=1. \tag{25}$$

However, as the principal stretches are functions of I_1 and I_2, it is clear that both approaches are consistent.

In [40], a large number of expressions of $\hat{W}(I_1, I_2)$ are reviewed. Let

$$\hat{I}_1 = I_1 - (I_3 - 1) \tag{26}$$

$$\hat{I}_2 = I_2 - 2(I_3 - 1) \tag{27}$$

and consider the function

$$W(I_1, I_2, I_3) = \hat{W}(\hat{I}_1, \hat{I}_2) + \chi F(I_3) \tag{28}$$

where χ is a parameter to be defined, and where $F(I_3)$ is subjected to the conditions

$$F(1) = \frac{dF(1)}{dI_3} = 0; \quad \frac{dF(1)}{dI_3^2} \neq 0, \tag{29}$$

$W(I_1, I_2, I_3)$ is characterized by the following properties:

(a) The corresponding Piola–Kirchhoff stresses S_{ij} vanish in the initial (undeformed, unstressed) configuration Γ of the body.

(b) It may be used to represent approximately the behaviour of an incompressible solid, the isochoric deformation potential of which is $\hat{W}(I_1, I_2)$.

The proofs are straightforward. Let

$$K_I = \frac{\partial \hat{W}}{\partial I_1}; \quad K_2 = \frac{\partial \hat{W}}{\partial I_2} \tag{30}$$

$$\Phi = 2\frac{\partial W}{\partial I_1} = 2K_1(\hat{I}_1, \hat{I}_2) \tag{31}$$

$$\Psi = 2\frac{\partial W}{\partial I_2} = 2K_2(\hat{I}_1, \hat{I}_2) \tag{32}$$

$$p = 2\frac{\partial W}{\partial I_3} = -2K_1(\hat{I}_1, \hat{I}_2) - 4K_2(\hat{I}_1, \hat{I}_2) + 2\chi\frac{dF(I_3)}{dI_3}. \tag{33}$$

Then, following [41], the stresses are given by

$$S_{ij} = \Phi\delta_{ij} + \Psi(I_1\delta_{ij} - G_{ij}) + pI_3G_{ij}^{-1}, \tag{34}$$

this is

$$\boxed{\begin{aligned} S_{ij} &= 2(\delta_{ij} - I_3G_{ij}^{-1})\cdot K_1(\hat{I}_1, \hat{I}_2) \\ &\quad + 2(I_1\delta_{ij} - G_{ij} - 2I_3G_{ij}^{-1})\cdot K_2(\hat{I}_1, \hat{I}_2) \\ &\quad + 2\chi I_3G_{ij}^{-1}\frac{dF(I_3)}{dI_3} \end{aligned}} \tag{35}$$

where G_{ij} is the metric tensor, G_{ij}^{-1} its inverse and δ_{ij} the Kronecker delta.

In the initial configuration Γ, we have

$$G_{ij} = G_{ij}^{-1} = \delta_{ij} \tag{36}$$

$$I_1 = I_2 = 3; \quad I_3 = 1. \tag{37}$$

Using (29), (36) and (37), it is seen that $S_{ij} = 0$ in the initial configuration. On the other hand, for an incompressible material, $I_3 = 1$ for any state of deformation. In this case, it is easily verified that $W(I_1, I_2, I_3 = 1)$ is identical to $\hat{W}(I_1, I_2)$. This demonstrates the second property.

3.2 *Incremental constitutive equation*

The elasticity coefficients C_{ijkl} appearing in the incremental equation

$$\mathrm{d}S_{ij} = C_{ijkl}\,\mathrm{d}E_{kl} \tag{38}$$

where E_{kl} is the Green's strain tensor, are computed hereafter. It can be shown that

$$\left.\begin{aligned} \mathrm{d}I_1 &= 2\delta_{kl}\,\mathrm{d}E_{kl}\\ \mathrm{d}I_2 &= 2(I_1\delta_{kl} - G_{kl})\,\mathrm{d}E_{kl}\\ \mathrm{d}I_3 &= 2I_3G_{kl}^{-1}\,\mathrm{d}E_{kl}\\ \mathrm{d}G_{ij}^{-1} &= -2G_{ik}^{-1}G_{jl}^{-1}\,\mathrm{d}E_{kl}\\ \mathrm{d}(I_3G_{ij}^{-1}) &= 2I_3(G_{ij}^{-1}G_{kl}^{-1} - G_{ik}^{-1}G_{jl}^{-1})\,\mathrm{d}E_{kl} \end{aligned}\right\} \tag{39}$$

so that the differential of (35) writes:

$$\left.\begin{aligned} \mathrm{d}S_{ij} = &-2\,\mathrm{d}(I_3G_{ij}^{-1})\cdot K_1 + 2(\delta_{ij} - I_3G_{ij}^{-1})\cdot \mathrm{d}K_1\\ &+ 2\,\mathrm{d}(I_1\delta_{ij} - G_{ij} - 2I_3G_{ij}^{-1})\cdot K_2 + 2(I_1\delta_{ij} - G_{ij} - 2I_3G_{ij}^{-1})\,\mathrm{d}K_2\\ &+ 2\cdot\chi\cdot\frac{\mathrm{d}F}{\mathrm{d}I_3}\cdot \mathrm{d}(I_3G_{ij}^{-1}) + 2\cdot\chi\cdot\frac{\mathrm{d}^2F}{\mathrm{d}I_3^2}I_3G_{ij}^{-1}\,\mathrm{d}I_3. \end{aligned}\right\} \tag{40}$$

Let

$$f_{11} = \frac{\partial^2 W}{\partial I_1^2};\quad f_{12} = \frac{\partial^2 W}{\partial I_1 \partial I_2};\quad f_{22} = \frac{\partial^2 W}{\partial I_2^2} \tag{41}$$

$$\left.\begin{aligned} \mathrm{d}K_1 &= f_{11}\cdot(\mathrm{d}I_1 - \mathrm{d}I_3) + f_{12}\cdot(\mathrm{d}I_2 - 2\,\mathrm{d}I_3)\\ \mathrm{d}K_2 &= f_{12}\cdot(\mathrm{d}I_1 - \mathrm{d}I_3) + f_{22}\cdot(\mathrm{d}I_2 - 2\mathrm{d}I_3). \end{aligned}\right\} \tag{42}$$

By introducing (41), (42) and (39) in (40), the following expression is obtained for C_{ijkl}

$$\begin{aligned} C_{ijkl} = &-4I_3(G_{ij}^{-1}G_{kl}^{-1} - G_{ik}^{-1}G_{jl}^{-1})\cdot K_1(\hat{I}_1, \hat{I}_2)\\ &+ 4[(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl}) - 2(G_{ij}^{-1}G_{kl}^{-1} - G_{ik}^{-1}G_{jl}^{-1})]\cdot K_2(\hat{I}_1, \hat{I}_2)\\ &+ 4I_3(G_{ij}^{-1}G_{kl}^{-1} - G_{ik}^{-1}G_{jl}^{-1})\cdot\chi\cdot\frac{\mathrm{d}F(I_3)}{\mathrm{d}I_3}\\ &+ 4(\delta_{ij} - I_3G_{ij}^{-1})\cdot[(\delta_{kl} - I_3G_{kl}^{-1})\cdot f_{11}(\hat{I}_1, \hat{I}_2)\\ &+ (I_1\delta_{kl} - G_{kl} - 2I_3G_{kl}^{-1})\cdot f_{12}(\hat{I}_1, \hat{I}_2)]\\ &+ 4(I_1\delta_{ij} - G_{ij} - 2I_3G_{ij}^{-1})[(\delta_{kl} - I_3G_{kl}^{-1})\cdot f_{12}(\hat{I}_1, \hat{I}_2)\\ &+ (I_1\delta_{kl} - G_{kl} - 2I_3G_{kl}^{-1})\cdot f_{22}(\hat{I}_1, \hat{I}_2)]\\ &+ 4I_3G_{ij}^{-1}G_{kl}^{-1}\cdot\chi\cdot\frac{\mathrm{d}^2F(I_3)}{\mathrm{d}I_3^2}. \end{aligned} \tag{43}$$

If these coefficients are evaluated in the initial configuration (Γ) using (29), (36) and (37) they become:

$$C_{ijkl} = -4[(K_1)_\Gamma + (K_2)_\Gamma]\cdot(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl}) + 4\cdot\chi\cdot\left(\frac{\mathrm{d}^2F}{\mathrm{d}I_3^2}\right)_\Gamma \delta_{ij}\delta_{kl} \tag{44}$$

where $(K_1)_\Gamma$, $(K_2)_\Gamma$ and $(\mathrm{d}^2F/\mathrm{d}I_3^2)_\Gamma$ are the values of K_1, K_2 and $(\mathrm{d}^2F/\mathrm{d}I_3^2)$ in the configuration Γ.

In order to identify the parameters appearing in (43), this result is compared with Hooke's tensor which writes

$$\left.\begin{aligned} C_{ijkl}^{\text{Hooke}} &= 2G\left(\delta_{ik}\delta_{jl} - \frac{\nu}{1-2\nu}\,\delta_{ij}\delta_{kl}\right) \\ &= -2G(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl}) + 2G\,\frac{1-\nu}{1-2\nu}\,\delta_{ij}\delta_{kl}. \end{aligned}\right\} \tag{45}$$

It is seen that

$$G = 2[(K_1)_\Gamma + (K_2)_\Gamma] \tag{46}$$

$$4\cdot\chi\cdot\left(\frac{d^2F}{dI_3^2}\right)_\Gamma = 2G\,\frac{1-\nu}{1-2\nu} = 4[(K_1)_\Gamma + (K_2)_\Gamma]\,\frac{1-\nu}{1-2\nu} \tag{47}$$

from where

$$\nu = \frac{4\cdot\chi\cdot\left(\frac{d^2F}{dI_3^2}\right)_\Gamma - 4[(K_1)_\Gamma + (K_2)_\Gamma]}{8\cdot\chi\cdot\left(\frac{d^2F}{dI_3^2}\right)_\Gamma - 4[(K_1)_\Gamma + (K_2)_\Gamma]}. \tag{48}$$

Thus

$$\lim_{\chi\to\infty} \nu = 0.5. \tag{49}$$

This shows that the parameter χ may be regarded as a bulk modulus. As for (46), it is a well-known result (see, e.g. [42]).

For a compressible material, the function $F(I_3)$ should be experimentally determined. But, in the case of nearly incompressible solids, the χ coefficient will always be very high and therefore, the particular form of $F(I_3)$ will be of negligible influence.

It is proposed to retain the simplest form, this is

$$\frac{d^2F}{dI_3^2} = 1 \quad \frac{dF}{dI_3} = I_3 - 1 \quad F = \frac{1}{2}(I_3 - 1)^2. \tag{50}$$

3.3 *A particular case: plane state of stress in a hyperelastic membrane*

In order to check the validity of the preceding developments, they are applied to the plane state of stress in a hyperelastic membrane. For this case, one has

$$\underset{\sim}{G} = \begin{bmatrix} G_{11} & G_{12} & 0 \\ G_{21} & G_{22} & 0 \\ 0 & 0 & \Lambda^2 \end{bmatrix}; \quad I_3\underset{\sim}{G}^{-1} = \begin{bmatrix} \Lambda^2 G_{22} & -\Lambda^2 G_{12} & 0 \\ -\Lambda^2 G_{21} & \Lambda^2 G_{11} & 0 \\ 0 & 0 & \frac{I_3}{\Lambda^2} \end{bmatrix}$$

$$I_1 = G_{11} + G_{22} + \Lambda^2; \qquad I_3 = \Lambda^2(G_{11}G_{22} - G_{12}G_{21})$$

where Λ is the thickness ratio of the membrane in the deformed and undeformed configurations.

For the sake of simplicity, it is assumed that the material obeys to Mooney's constitutive law [43]

$$\hat{W} = C_1\cdot(I_1 - 3) + C_2\cdot(I_2 - 3). \tag{51}$$

Then, using (50), (28) writes

$$W(I_1, I_2, I_3) = C_1\cdot(I_1 - I_3 - 2) + C_2\cdot(I_2 - 2I_3 - 1) + \frac{1}{2}\cdot\chi\cdot(I_3 - 1)^2. \tag{52}$$

Applying (35), one obtains

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = 2C_1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2C_2\begin{bmatrix} G_{22}+\Lambda^2 & -G_{12} \\ -G_{21} & G_{11}+\Lambda^2 \end{bmatrix} + 2\Lambda^2[\chi(I_3-1) - C_1 - 2C_2]\begin{bmatrix} G_{22} & -G_{12} \\ -G_{21} & G_{11} \end{bmatrix} \tag{53}$$

$$S_{33} = 2C_1 + 2C_2(G_{11} + G_{22}) + 2\frac{I_3}{\Lambda^2}[\chi(I_3 - 1) - C_1 - 2C_2]. \tag{54}$$

But, by hypothesis, $S_{33} = 0$. From (54), it is seen that

$$2I_3[\chi(I_3 - 1) - C_1 - 2C_2] = -\Lambda^2[2C_1 + 2C_2(G_{11} + G_{22})]. \tag{55}$$

Introducing this equation in (53) one finds:

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = 2C_1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2C_2\begin{bmatrix} G_{22}+\Lambda^2 & -G_{12} \\ -G_{21} & G_{11}+\Lambda^2 \end{bmatrix} - 2\frac{\Lambda^4}{I_3}[(C_1 + (G_{11} + G_{22})C_2]\begin{bmatrix} G_{22} & -G_{12} \\ -G_{21} & G_{11} \end{bmatrix}. \tag{56}$$

This result is identical to that given by Oden[29, form. (18.23), p. 302] in the case of an incompressible material.

Furthermore, the Piola–Kirchhoff stresses S^h_{ij} corresponding to a true hydrostatic pressure h are

$$S^h_{ij} = hG^{-1}_{ij}. \tag{57}$$

Consequently, in the particular case examined here, the true hydrostatic pressure is

$$h = -2\Lambda^2[C_1 + (G_{11} + G_{22})C_2] = 2I_3[\chi(I_3 - 1) - C_1 - 2C_2]. \tag{58}$$

4. EXTENSION OF NAGTEGAAL'S PRINCIPLE TO NONLINEAR ELASTICITY

The fundamental idea of Nagtegaal's principle in linear analysis is to discretize the volumetric dilatation independently of the displacement field. The same idea may be applied in nonlinear elasticity by discretizing separately the quantity

$$\theta = I_3 - 1. \tag{59}$$

For a nearly incompressible hyperelastic material, the constitutive equation is, from (28) and (50)

$$W(I_1, I_2, I_3) = \hat{W}(\hat{I}_1, \hat{I}_2) + \frac{1}{2}\cdot\chi\cdot(I_3 - 1)^2. \tag{60}$$

Consider now the functional

$$H(\theta, U_i) = \int_V \left\{\chi\left[\theta(I_3 - 1) - \frac{1}{2}\theta^2\right] + \hat{W}(\hat{I}_1, \hat{I}_2) - \rho^0 g_i U_i\right\} \mathrm{d}V - \int_{A_T} T_i U_i \,\mathrm{d}A_T \tag{61}$$

where U_i is the large displacement field in total Lagrangian description (TLD); ρ^0 is the mass density (per unit undeformed volume); g_i is the distributed body force per unit mass; T_i is the surface traction per unit undeformed area; V is the volume of the body in the undeformed configuration Γ; and A_T is that part of the undeformed surface of the body where surface tractions T_i are applied.

It is easily verified that the above functional leads to the nonlinear equilibrium equations and to eqn (59) by taking the variations on U_i and θ. The variation on U_i gives

$$\delta_{U_i} H = \int_V \{(\sigma G^{-1}_{ij} + \hat{S}_{ij})\delta E_{ij} - \rho^0 g_i \delta U_i\}\,\mathrm{d}V - \int_{A_T} T_i \delta U_i \,\mathrm{d}A_T = 0.$$

After integrating by parts, one obtains

$$\frac{\partial}{\partial X_j}[(\hat{S}_{jk} + \sigma G^{-1}_{jk})(\delta_{ik} + U_{ik})] + \rho^0 g_i = 0 \qquad \text{in } V \tag{62}$$

$$N_j[(\hat{S}_{jk} + \sigma G^{-1}_{jk})(\delta_{ik} + U_{ik})] = T_i \qquad \text{on } A_T \tag{63}$$

where

$$\hat{S}_{ij} = 2(\delta_{ij} - I_3 G_{ij}^{-1}) \cdot K_1(\hat{I}_1, \hat{I}_2) + 2(I_1\delta_{ij} - G_{ij} - 2I_3 G_{ij}^{-1}) \cdot K_2(\hat{I}_1, \hat{I}_2) \tag{64}$$

$$\sigma = 2I_3 \cdot \chi \cdot \theta. \tag{65}$$

On the other hand, the variation on θ immediately gives (59).

By introducing this equation in (62) and (63), the classical equilibrium equations are recovered. This demonstrates the validity of the proposed functional. Furthermore, it is seen that the stresses are to be computed by

$$S_{ij} = \hat{S}_{ij} + \sigma G_{ij}^{-1}. \tag{66}$$

This formula shows that $\hat{S}_{ij}$ may be considered as a deviatoric stress

$$\hat{S}_{ij} = \frac{\partial \hat{W}(\hat{I}_1, \hat{I}_2)}{\partial E_{ij}} \tag{67}$$

while σ appears as a true hydrostatic stress.

However, σ is only a part of the hydrostatic pressure which can be computed by

$$\frac{h}{2I_3} = \frac{\partial W}{\partial I_3} \tag{68}$$

this is

$$h = 2I_3[\chi(I_3 - 1) - K_1 - 2K_2] = \sigma - 2I_3(K_1 + 2K_2). \tag{69}$$

Returning to the formation of the constitutive law (28), one realizes that the term

$$-2I_3(K_1 + 2K_2)$$

has been introduced to make the stresses vanish in the undeformed configuration Γ. Thus, it exists before any deformation of the body. On the other hand, σ vanishes in Γ. It may therefore be considered as the hydrostatic stress induced by the deformation of the body, especially in materials for which K_1 and K_2 remain constant.

5. CONNECTION WITH HELLINGER-REISSNER'S PRINCIPLE

The functional (61) can be deduced from Hellinger–Reissner's

$$F_{HR}(S_{ij}, U_i) = \int_V \left\{ \frac{1}{2} S_{ij}(U_{i,j} + U_{j,i} + U_{k,i}U_{k,j}) - S(S_{ij}) - \rho^0 g_i U_i \right\} \mathrm{d}V - \int_{A_T} T_i U_i \,\mathrm{d}A_T$$
$$- \int_{A_U} N_j S_{jk}(\delta_{ik} + U_{i,k})(U_i - \bar{U}_i) \,\mathrm{d}A_U \tag{70}$$

where $\bar{U}_i$ denotes the imposed displacements on A_U, and $S(S_{ij})$ is the complementary energy density. The following hypothesis are introduced:

$$U_i = \bar{U}_i \qquad \text{on } A_u \tag{71}$$

$$S_{ij} = \sigma G_{ij}^{-1} \qquad \text{in } V \tag{72}$$

$$\sigma = 2I_3 \cdot \chi \cdot (I_3 - 1). \tag{73}$$

Then, one has successively

$$W(I_1, I_2, I_3) = \hat{W}(\hat{I}_1, \hat{I}_2) + \frac{1}{2\chi}\left(\frac{\sigma}{2I_3}\right)^2 \tag{74}$$

$$S_{ij}E_{ij} = \sigma G_{ij}^{-1} E_{ij} = \frac{1}{2}\sigma G_{ij}^{-1}(G_{ij} - \delta_{ij}).$$

But, it is easily verified that

$$G_{ij}^{-1}G_{ij} = 3; \qquad G_{ij}^{-1}\delta_{ij} = \frac{I_2}{I_3}.$$

So

$$S_{ij}E_{ij} = \frac{\sigma}{2I_3}(3I_3 - I_2) = \frac{\sigma}{2I_3}(I_3 - 1) - \frac{\sigma}{2I_3}(I_2 - 2I_3 - 1).$$

Using (73), this becomes

$$S_{ij}E_{ij} = \frac{1}{\chi}\left(\frac{\sigma}{2I_3}\right)^2 - \frac{\sigma}{2I_3}(I_2 - 2I_3 - 1).$$

Finally, the complementary energy density S writes

$$S = \frac{1}{2\chi}\left(\frac{\sigma}{2I_3}\right)^2 - \hat{W}(\hat{I}_1, \hat{I}_2) - \frac{\sigma}{2I_3}(I_2 - 2I_3 - 1). \tag{75}$$

On the other hand

$$\frac{1}{2}S_{ij}(U_{i,j} + U_{j,i} + U_{k,i}U_{k,j}) = \sigma G_{ij}^{-1}E_{ij} = \frac{\sigma}{2I_3}(I_3 - 1) - \frac{\sigma}{2I_3}(I_2 - 2I_3 - 1). \tag{76}$$

With (71), (75) and (76), (70) writes:

$$H(\sigma, U_i) = \int_V \left\{\left(\frac{\sigma}{2I_3}\right)(I_3 - 1) - \frac{1}{2\chi}\left(\frac{\sigma}{2I_3}\right)^2 + \hat{W}(\hat{I}_1, \hat{I}_2) - \rho^0 g_i U_i\right\} \mathrm{d}V - \int_{A_T} T_i U_i \,\mathrm{d}A_T. \tag{77}$$

This generalizes Key's functional (7) to the nonlinear domain.

Taking the variation on σ, it is found:

$$\frac{\sigma}{2I_3} = \chi(I_3 - 1) = \chi\theta. \tag{78}$$

The introduction of this result in (77) gives (61).

The connection of both functionals (61) and (77) with Hellinger–Reissner's principle is now established.

6. SINGLE FIELD PRINCIPLES, INTERMEDIATE PRINCIPLE

In (61) assume that

$$\theta = \theta_0 = \text{constant in } V. \tag{79}$$

The variation on θ_0 gives

$$V\theta_0 = \int_V (I_3 - 1)\,\mathrm{d}V. \tag{80}$$

Introducing (80) in (61), one obtains

$$H(U_i) = \int_V \left\{W(I_1, I_2, I_3) + \frac{1}{2}\cdot\chi\cdot\left[\left(\frac{1}{V}\int_V (I_3 - 1)\,\mathrm{d}V\right)^2 - (I_3 - 1)^2\right] - \rho^0 g_i U_i\right\}\mathrm{d}V - \int_{A_T} T_i U_i \,\mathrm{d}A_T. \tag{81}$$

This form differs from the minimum energy principle only by the second term in the volume integral which takes into account the difference between the true volumetric dilatation energy $\frac{1}{2}\cdot\chi\cdot(I_3 - 1)^2$ and its average value $\frac{1}{2}\cdot\chi\cdot(1/V\int_V(I_3 - 1)\,\mathrm{d}V)^2$ over the volume V.

By analogy with (21) an intermediate principle may be introduced by multiplying this term by a coefficient m:

$$H(U_i)=\int_V\left\{W(I_1,I_2,I_3)+\frac{1}{2}\chi m\left[\left(\frac{1}{V}\int_V(I_3-1)\,\mathrm{d}V\right)^2-(I_3-1)^2\right]-\rho^0 g_i U_i\right\}\mathrm{d}V-\int_{A_T}T_iU_i\,\mathrm{d}A_T. \tag{82}$$

In particular, m may write

$$m=1-(1-\alpha)^2\cdot\frac{4[K_1(\hat{I}_1,\hat{I}_2)+K_2(\hat{I}_1,\hat{I}_2)]}{12\chi-8[K_1(\hat{I}_1,\hat{I}_2)+K_2(\hat{I}_1,\hat{I}_2)]}. \tag{83}$$

In this formula, α contributes in the same way as in the intermediate principle introduced in linear analysis. Furthermore, with the help of (46) and (48) it is seen that in the initial configuration Γ one has:

$$m=1-(1-\alpha)^2\frac{1-2\nu}{1+\nu}. \tag{84}$$

This completes the analogy.

Taking $\alpha=0$, the functional (82) becomes

$$H(U_i)=\int_V\left\{W(I_1,I_2,I_3)+\frac{1}{2}\chi\left[1-\frac{4(K_1+K_2)}{12\chi-8(K_1+K_2)}\right]\cdot\left[\left(\frac{1}{V}\int_V(I_3-1)\,\mathrm{d}V\right)^2-(I_3-1)^2\right]\right.$$
$$\left.-\rho^0g_iU_i\right\}\mathrm{d}V-\int_{A_T}T_iU_i\,\mathrm{d}A_T \tag{85}$$

and may be considered as an extension of Herrmann's principle (4).

7. USE OF MODIFIED INVARIANTS AND COMPUTATION OF STRESSES

The variation on U_i in (81) gives the following equilibrium equations:

$$\begin{aligned}&\frac{\partial}{\partial X_j}[(\hat{S}_{jk}+S^0_{jk})\cdot(\delta_{ik}+U_{i,k})]+\rho^0g_i=0 \qquad \text{in } V\\ &N_j[(\hat{S}_{jk}+S^0_{jk})\cdot(\delta_{ik}+U_{i,k})]=T_i \qquad \text{on } A_T\end{aligned} \tag{86}$$

where N_j is the outer normal to A_T and

$$S^0_{jk}=\chi\left[\frac{1}{V}\int_V(I_3-1)\,\mathrm{d}V\right]\cdot 2I_3G^{-1}_{jk}. \tag{87}$$

This shows that the stresses deriving from the functional (81) are to be computed by

$$S_{ij}=\hat{S}_{ij}+S^0_{ij}. \tag{88}$$

By analogy with the case of linear analysis, it is interesting to put (81) in the form

$$H(U_i)=\int_V\{W(I^*_1,I^*_2,I^*_3)-\rho^0g_iU_i\}\,\mathrm{d}V-\int_{A_T}T_iU_u\,\mathrm{d}A_T. \tag{89}$$

This is done by defining the following modified invariants

$$\begin{aligned}I^*_1&=I_1+\ [\theta_0-(I_3-1)]\\ I^*_2&=I_2+2[\theta_0-(I_3-1)]\\ I^*_3&=I_3+\ [\theta_0-(I_3-1)]=\theta_0+1.\end{aligned} \tag{90}$$

One has

$$\hat{I}_1^* = \overset{*}{I}_1 - \overset{*}{I}_3 = I_1 - I_3 = \hat{I}_1$$
$$\hat{I}_2^* = \overset{*}{I}_2 - 2\overset{*}{I}_3 = I_2 - 2I_3 = \hat{I}_2 \tag{91}$$

consequently

$$W(\overset{*}{I}_1, \overset{*}{I}_2, \overset{*}{I}_3) = \hat{W}(\hat{I}_1^*, \hat{I}_2^*) + \frac{1}{2}\cdot\chi\cdot(\overset{*}{I}_3 - 1)^2 = \hat{W}(\hat{I}_1, \hat{I}_2) + \frac{1}{2}\cdot\chi\cdot\left[\frac{1}{V}\int_V (I_3 - 1)\,\mathrm{d}V\right]^2 \tag{92}$$

which demonstrates the identity of (81) and (89).

Furthermore, the stresses computed by

$$S_{ij} = \frac{\partial W(\overset{*}{I}_1, \overset{*}{I}_2, \overset{*}{I}_3)}{\partial E_{ij}} \tag{93}$$

may be shown to be given by (see[4])

$$S_{ij} = \hat{S}_{ij} + \bar{S}_{ij}^0 \tag{94}$$

where

$$\bar{S}_{ij}^0 = 2\frac{\chi}{V}\left[\frac{1}{V}\int_V (I_3 - 1)\,\mathrm{d}V\right]\int_V I_3 G_{ij}^{-1}\,\mathrm{d}V. \tag{95}$$

They differ from (87), (88), by the fact that S_{ij}^0 is replaced by its average value $\bar{S}_{ij}^0$ over the element. In a refined finite element mesh, the difference should not be important.

8. NUMERICAL EXAMPLE

Coinsider the problem of the inflation of a hollow sphere under internal pressure p (Fig. 1). In the initial configuration Γ, the dimensions are $R_a = 20$; $R_b = 30$.

It is assumed that the material obeys Mooney's law (51, 52), with $C_1 = 10$ and $C_2 = 1$. Several values of χ are tested: $\chi = 100$; 1000; 10,000; 100,000. A closed-form solution is given in[41] for an incompressible material and will be used to check the validity of the method proposed in the preceding paragraphs.

For the finite element analysis, the sphere is divided into five equally thick concentric spherical elements (Fig. 2).

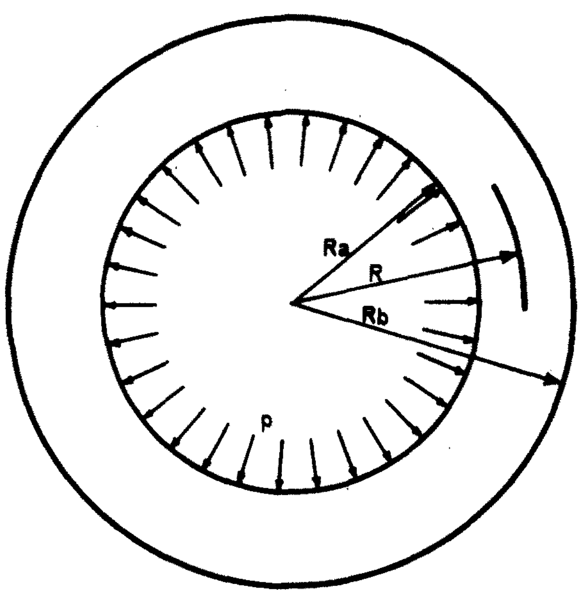

Fig. 1. Hollow sphere under internal pressure. Initial configuration Γ.

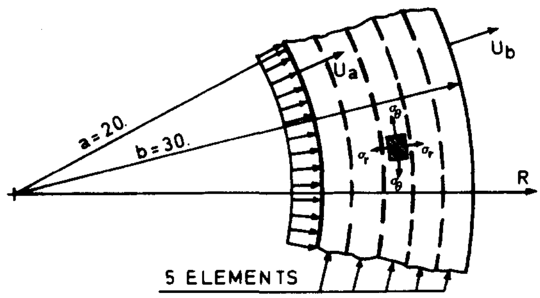

Fig. 2. Discretization.

Let R_1 and R_2 denote the internal and external radii of an element in the configuration Γ. The displacement field will be approximated by a linear function

$$U(R) = \frac{1}{R_2 - R_1}[(R_2 - R)U_1 + (R - R_1)U_2]$$

while the volumetric dilatation θ will be assumed to be constant over each element.

The details of the development may be found in[4] and the results are given in Figs. 3–5 and Table 1.

Table 1.

Theory		Finite Elements							
		$\chi = 100$		$\chi = 1000$		$\chi = 10,000$		$\chi = 100,000$	
p	U_a	U_a	NCORR	U_a	NCORR	U_a	NCORR	U_a	NCORR
4.90474	2.0	2.052	2	2.004	3	1.999	3	1.999	3
7.92782	4.0	4.117	2	4.008	3	3.997	4	3.996	4
9.77585	6.0	6.205	2	6.012	3	5.994	5	5.992	5
10.87679	8.0	8.336	2	8.019	3	7.989	5	7.986	5
11.94989	10.0	10.556	2	10.031	3	9.981	6	9.976	6

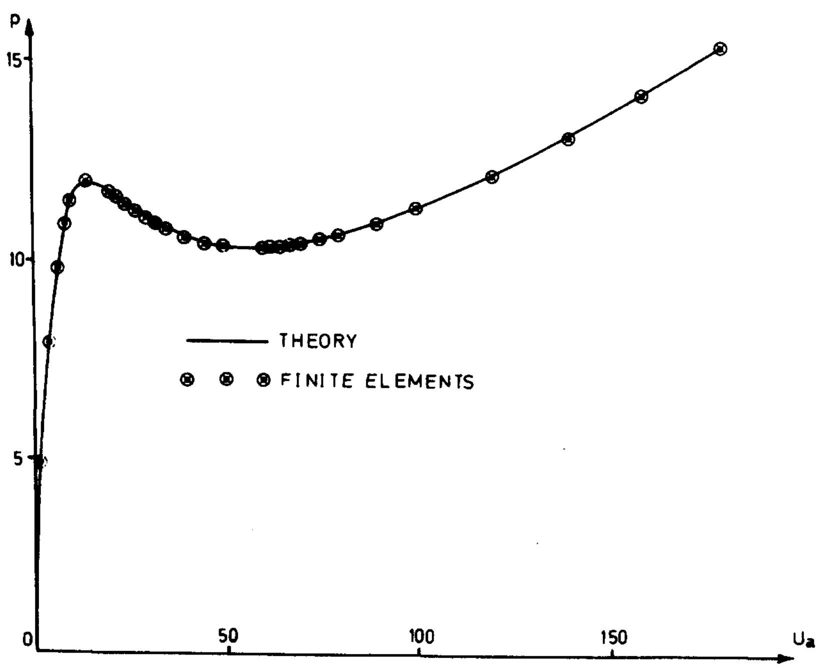

Fig. 3. Displacement U_a of the internal surface.

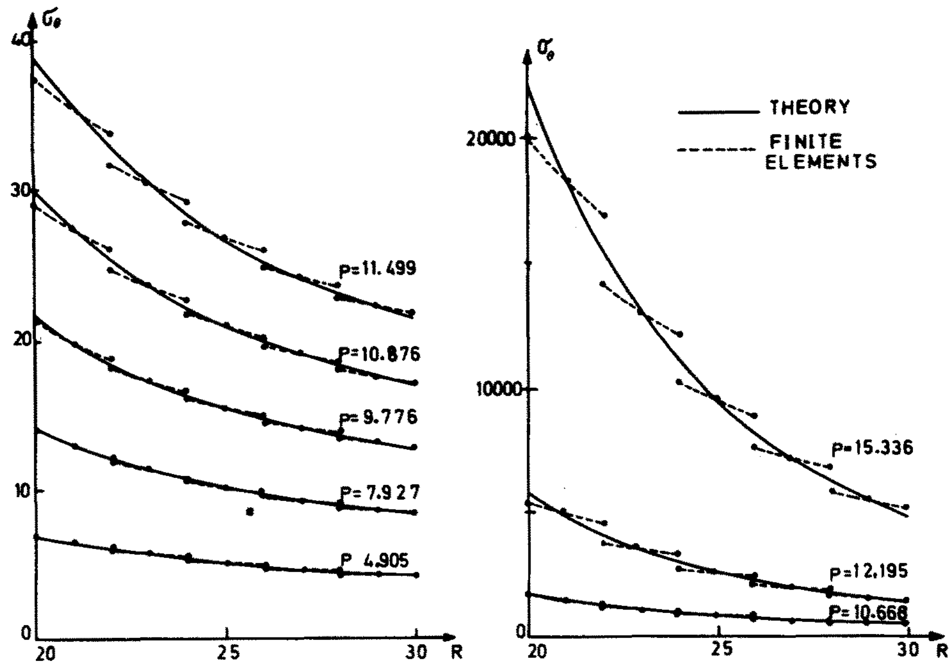

Fig. 4. Azimuthal Cauchy stresses σ_θ.

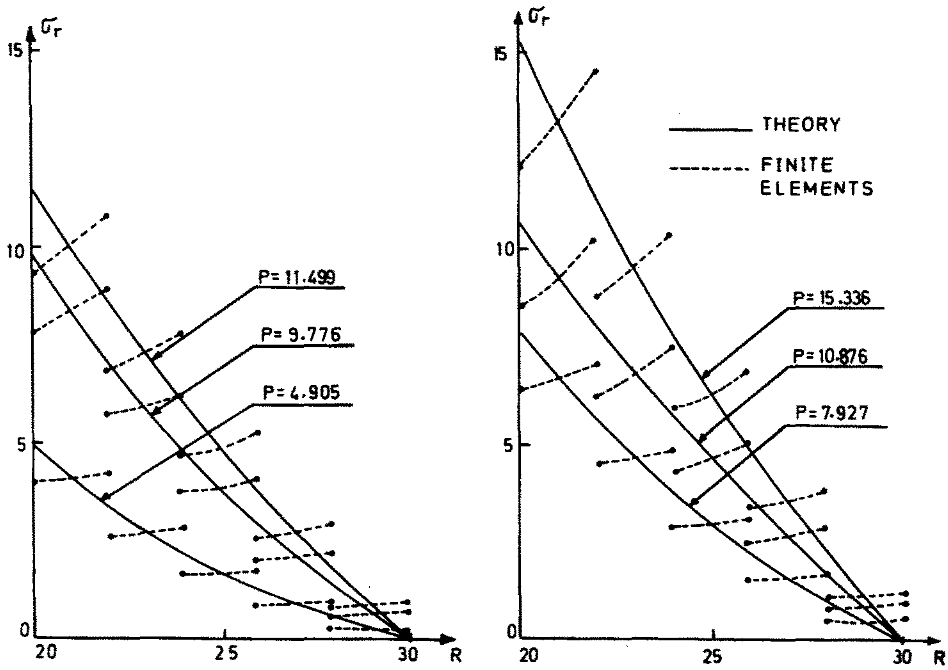

Fig. 5. Radial Cauchy stresses σ_r.

The complete load displacement curve is computed for $\chi = 100{,}000$ (Fig. 3; step by step solution with Newton–Raphson corrections). It agrees remarkably well with the theoretical solution.

In Table 1, results are compared for several values of χ. It is seen that they converge towards the exact solution as χ is increased, but the number of Newton–Raphson corrections (NCORR) at each step also increases. (This fact has been mentioned by Skala[37].) Consequently it is necessary to compromise between the desired accuracy of the solution and the computer cost.

The stresses are given in Figs. 4 and 5. They also agree with the theory, even for very large deformations.

9. CONCLUSION

The functional (61) is an extension of Nagtegaal's principle (12) for hyperelastic nearly incompressible materials. It may be considered as a particular application of Hellinger–Reissner's variational principle. Its main advantages are:

(a) Except that the material should be hyperelastic, no particular form of the constitutive law has been assumed. Furthermore, if the material parameters are known for a given incompressible body, no additional experiment is required to use the correspondingly nearly incompressible constitutive equation (60).

(b) The discretization of the volumetric dilatation is free. In particular, the number of discretization parameters may be chosen so as to ensure good convergence to the exact solution when the material becomes more and more incompressible.

In other words, the variable θ allows to control the number of incompressibility constraints. This is not the case when the equivalence theorem of Malkus and Hughes is used. For example, in the 8-node serendipity element[50], the latter imposes 4 constraints per element. With (61), it is also possible to choose

$$\theta = \theta_0 + \theta_\xi \cdot \xi + \theta_\eta \cdot \eta$$

in local axes (ξ, η), with only 3 discretization parameters θ_0, θ_ξ, θ_η giving 3 constraints, or even

$$\theta = \theta_0$$

giving one single constraint per element.

(c) The use of modified invariants enables to put (61) in a remarkably simple form (89), similar to the minimum energy principle. This simplifies the use of this method to transform classical displacement-type finite elements[4].

Obviously, this approach can be interpreted as a penalty method[51, 52]. It requires the choice of a suitable value for the parameter χ. From an engineer viewpoint, this is not too difficult because this parameter has been connected with the Poisson's coefficient in (48). It is the author's opinion that

$$\chi = 1000[(K_1)_r + (K_2)_r]$$

(this corresponds to $\nu = 0.499$ in the infinitesimal case) is a good value. This is confirmed by the results in Table 1.

REFERENCES

1. L. R. Herrmann and R. M. Toms, A reformulation of the elastic field equations in terms of displacements, valid for all admissible values of Poisson's ratio. *Trans. ASME, J. Appl. Mech.* 140–141 (1964).
2. L. R. Herrmann, Elasticity equations for incompressible and nearly incompressible materials by a variational theorem. *AIAA J.* 3(10) (1965).
3. R. L. Taylor, K. S. Pister and L. R. Herrmann, On a variational theorem for incompressible and nearly incompressible orthotropic elasticity. *Int. J. Solids Structures* 4, 875–883 (1968).
4. S. Cescotto, Etude par éléments finis des grands déplacements et grandes déformations. Application aux problèmes spécifiques des matériaux quasi-incompressibles. Chap VI: Analyse infinitésimale des corps incompressibles on quasi incompressibles. Ph.D. Thesis Université de Liège (1978).
5. I. Fried, Influence of Poisson's ratio on the condition of the finite elements stiffness matrix. *Int. J. Solids Structures* 9, 323–329 (1973).
6. I. Fried, Finite element analysis of incompressible material by residual energy balancing. *Int. J. Solids Structures* 10 993–1002 (1974).
7. J. C. Nagetegaal, D. M. Parks, J. R. Rice, On numerically accurate finite element solutions in the fully plastic range. *Comp. Meth. Appl. Mech. Engng* 4, 153–177 (1974).
8. J. H. Argyris, P. C. Dunne, T. Angelopoulos, B. Bichat, Large natural strains and some special difficulties due to non linearity and incompressibility in finite elements. *Comp. Meth. Appl. Mech. Engng* 4, 219–278 (1974).
9. J. H. Argyris, P. C. Dunne, Th.L. Johnsen and M. Muller, Linear systems with a large number of sparse constraints with applications to incompressible materials. *Comp. Meth. Appl. Mech. Engng* 10, 105–132 (1977).
10. J. H. Argyris, D. C. Dunne and M. Muller, Isochoric constant strain finite elements. *Comp. Meth. Appl. Mech. Engng* 13(2) 245–278 (1978).
11. J. F. Debongnie, On the approximation of incompressible solids in the displacement method. Rapport SA-42 LTAS Université de Liège (Jan. 1976).
12. J. F. Debongnie, Sur la formulation de Herrmann pour l'étude des solides incompressibles. *J. de Mécanique* 17(4) (1978).

13. J. J. Too, Two dimension plate, shell and finite prism isoparametric elements and their applications. Ph.D. Thesis, University of Wales (Nov. 1971).
14. O. C. Zienkiewicz, R. L. Taylor and J. J. Too, Reduced integration technique in general analysis of plates and shells. *Int. Num. Meth. Engng* **3**, 275–290 (1971).
15. O. C. Zienkiewicz and P. N. Godbole, Viscous incompressible flow with special reference to non-Newtonian (plastic) fluids, In *Finite Element Method in Flow Problems*. Wiley, New York (1975).
16. D. J. Naylor, Stresses in nearly incompressible materials by finite elements with application to the calculation of excess pore pressure. *Int. J. Num. Meth. Engng* **8**, 443–460 (1974).
17. I. Fried, Numerical integration in the finite element method. *Comput. Struct.* **4**, 921–932 (1974).
18. D. S. Malkus, Finite element analysis of incompressible solids. Ph.D. Thesis, Boston University (1975).
19. D. S. Malkus, A finite element displacement model valid for any value of the compressibility. *Int. J. Solids Structures* **12**, 731–738 (1976),
20. T. J. R. Hughes, Equivalence of finite elements for nearly incompressible elasticity. *J. Appl. Mech.* **44**, 181–183 (1977).
21. T. J. R. Hughes and D. S. Malkus, On the equivalence of mixed finite element methods with reduced/selective integration displacement methods. *Proc. Symp. Applic. Computer Methods in Engng* Los Angeles (Aug. 1977).
22. D. S. Malkus and T. J. R. Hughes, Mixed finite element methods—reduced and selective integration techniques. A unification of concepts. To be published.
23. S. W. Key, A variational principle for incompressible and nearly incompressible anisotropic elasticity. *Int. J. Solids Structures* **5**, 951–964 (1969).
24. E. G. Thompson and M. I. Haque, A high order finite element for completely incompressible creeping flow. *Int. J. Num. Meth. Engng* **6**, 315–321 (1973).
25. E. G. Thompson, Average and complete incompressibility in the finite element method. *Int. J. Num. Meth. Engng* **9**, 925–932 (1975).
26. P. Tong and T. H. H. Pian, A variational principle and the convergence of a finite element method based on assumed stress distribution. *Int. J. Solids Structures* **5**, 463–472 (1969).
27. P. Tong, An assumed stress hybrid finite element method for an incompressible and near-incompressible material. *Int. J. Solids Structures* **5**, 455–461 (1969).
28. R. W. Ogden, Volume changes associated with the deformation of rubber-like solids. *J. Mech. Phys. Solids* **24**, 323–338 (1976).
29. J. T. Oden, *Finite Elements of Nonlinear Continua*. McGraw-Hill, New York (1972).
30. J. T. Oden and J. E. Key, On some generalization of the incremental stiffness relations for finite deformations of compressible and incompressible finite elements. *Nucl. Engng Design* **15**, 121–134 (1971).
31. J. T. Oden, Numerical formulation of nonlinear elasticity problems. *Proc. ASCE ST.* **3** (June 1967).
32. J. T. Oden and J. E. Key, Numerical analysis of finite axisymmetric deformations of incompressible elastic solids of revolution. *Int. J. Solids Structures* **6**, 497–518 (1970).
33. J. T. Oden and J. E. Key, Analysis of finite deformations of elastic solids by the finite element method. *Proc. IUTAM Coll. High Speed Computing Elastic Structures*, Liège (1971).
34. T. Scharnhorst and T. H. H. Pian, Finite element analysis of rubber-like materials by a mixed model. *Int. J. Num. Meth. Engng* **12**, 665–676 (1978).
35. P. L. Boland and T. H. H. Pian, Large deflection analysis of thin-elastic structures by the assumed stress hybrid finite element method. *Comput. Struct.* **7**, 1–12 (1977).
36. T. Scharnhorst and T. H. H. Pian, Incremental iterative solutions of rubber-like materials by a mixed model. Presented as a short contribution at the *Int. Conf. on Finite Elements in Nonlinear Solid and Structural Mechanics*, Geilo. Norway, 29 Aug.–1 Sept. (1977).
37. D. P. Skala, Large strain finite analysis for a compressible elastomeric material. *Proc. Symp. Appl. Comp. Meth. Engng*, Los Angeles (Aug. 1977).
38. P. B. Lindley, A finite element program for the plane strain analysis of rubber. *J. Strain Anal.* **10**(1), 25 (1975).
39. L. R. G. Treloar, *The Physics of Rubber Elasticity*, 3rd Edn. Clarendon Press, Oxford (1975).
40. S. Cescotto, Grandes déformations de matériaux hyperélastiques en état plan de contraintes par la méthode des éléments finis. *Collection des Publications de la Faculté des Sciences Appliquées* (No. 64), Université de Liège (1977).
41. A. E. Green and W. Zerna, *Theoretical Elasticity*, 2nd Edn. Oxford University Press, London (1968).
42. H. Alexander, A constitutive relation for rubber-like materials. *Int. J. Engng Sci.* **6**, 549–563 (1968).
43. M. Mooney, A theory of large elastic deformation. *J. Appl. Phys.* **11**, 582–592 (1940).
44. T. J. Hughes, R. L. Taylor and J. L. Sackman, Finite element formulation and solution of contact and impact problems in continuum mechanics-III. Report 75-7, Berkeley, University of California (July 1975).
45. J. T. Oden, Analysis of large deformations of elastic membranes by the finite element method. *Proc. IASS Int. Cong. Large Span Shells*, Leningrad (1966).
46. J. T. Oden and W. K. Kubitza, Numerical analysis of nonlinear pneumatic structures. *Proc. Int. Coll. Pneumatic Structures*, Stuttgart 82–107 (May, 1967).
47. K. J. Bathe, H. Ozdemir and E. L. Wilson, Static and Dynamic Geometric and Material Non Linear Analysis. Report No. UC-SESM 74-4, University of California, Berkeley (1974).
48. K. J. Bathe, E. Ramm and E. L. Wilson, Finite element formulations for large deformation dynamic analysis. *Int. J. Num. Meth. Engng* **9**, 353–386 (1975).
49. J. W. Leonard and V. K. Verma, Double-curved element for Mooney–Rivlin membranes. *ASME EM* **4**, 625–641 (Aug. 1976).
50. O. C. Zienkiewicz, *The Finite Element Method in Engineering Science*. McGraw-Hill, New York (1971).
51. G. Strang and G. Fix, *An Analysis of the Finite Element Method*. Prentice Hall, Englewood Cliffs, New Jersey (1973).
52. R. Courant, Variational methods for the solution of problems of equilibrium and vibrations. *Bull. Am. Math. Soc.* **49**, 1–23.